On Hypermomentum in General Relativity
I. The Notion of Hypermomentum

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In this series of notes, we introduce a new quantity into the theory of classical matter fields. Besides the usual energy-momentum tensor, we postulate the existence of a further dynamical attribute of matter, the 3rd rank tensor Δ^{ijk} of hypermomentum. Subsequently, a general relativistic field theory of energy-momentum and hypermomentum is outlined. In Part I we motivate the need for hypermomentum. We split it into spin angular momentum, the dilatation hypermomentum, and traceless proper hypermomentum and discuss their physical meanings and conservation laws.

1. Spacetime as a 4-Dimensional Elastic Continuum

The vacuum is a complicated physical entity: it can be polarized (the Lamb shift), some of its symmetries can be broken or break spontaneously, its metric field g_{ij} $(i, j, \ldots = 0, 1, 2, 3)$ represents the gravitational potential, and through it propagate all other physical fields. In many ways it seems to be at least as complex a construct as the mechanical aether of the last century. Spacetime is well represented by a mathematical continuum (the differential manifold ** X_4) at least down to the scale of 10^{-14} cm or so. This continuum, in the light of Einstein's theory of gravitation, the general theory of relativity (GR), is neither static nor rigid, but reacts elastically to its matter distribution. (Just as one does in GR, we shall always regard the matter distribution as foreign to the geometry of spacetime itself.)

Let us conduct a thought experiment, and distribute matter in a flat Minkowski spacetime R_4. According to GR, the flat R_4 deforms into the Riemannian spacetime V_4. The spacetime thus deformed becomes flat once more when we remove the matter, hence the conception of spacetime as a gravitationally elastic medium. Let the matter distribution be characterized by its Lagrangian density $\mathcal{L}$. Then Hilbert's definition of the energy-momentum tensor $e\,\sigma^{ij} := 2\,\delta\mathcal{L}/\delta g_{ij}$ is a mathematical expression of this elastic property of spacetime.

We begin, then, by accepting the notion of spacetime as a 4-dimensional differential manifold X_4 and by accepting the existence of dynamical definitions (like Hilbert's) as an expression of the elastic reaction of spacetime to an imposed matter distribution.

2. Hyperstress in 3-Dimensional Polar Continua

Consider first a 3-dimensional elastic continuum. Cut a very small cavity out of an ordinary elastic continuum and impose a prescribed stress distribution on the surface of the cavity (one could realize this physically by inserting a larger piece of foreign material into the cavity). It has long been known (see, for example, Love [2]) that, to a first approximation, the action of the stresses can be represented, when the net force vanishes, by double forces with and without moments. By means of these tools it is straightforward to build up a so-called "center of rotation" out of the double forces with moments, and a "center of dilatation" out of the double forces without moments. A point defect in a crystal represents a realization of a center of dilatation, for example. These notions can be sharpened to the definition of an elastic dipole.

Recall the definition of an electric dipole as a limiting case of the situation where two equal and opposite charges $+q$ and $-q$ are separated by a distance Δx^α. The distance Δx^α tends to zero, but the charges are increased so that the product $|q|\cdot|\Delta x^\alpha|$ remains constant throughout the process. In an exact analogy, let us now imagine two equal and opposite forces $+f^\beta$ and $-f^\beta$ whose points of application are

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** Our mathematical notation is taken from Schouten [1]. $e := (\det g_{ij})^{1/2}$. Symmetrization and antisymmetrization of indices are denoted by () and [], respectively. $i, j \ldots = 0, 1, 2, 3$; $\alpha, \beta \ldots = 1, 2, 3$. δ=variational derivative, ∇_k = covariant derivative.

separated by the vector Δx^α. Then we define the elastic dipole moment according to

$$D^{\alpha\beta} := \lim_{\Delta x \to 0} \Delta x^\alpha f^\beta \qquad (\Delta x\, f = \text{const}) \,. \quad (1)$$

Here and hereafter, we regard these elastic dipoles as intrinsic objects, not reducible into pairs of forces. Then we call $D^{\alpha\beta}$ a "hyperforce" (see Jaunzemis [3], Kröner [4], Truesdell-Toupin [5], Truesdell-Noll [6]).

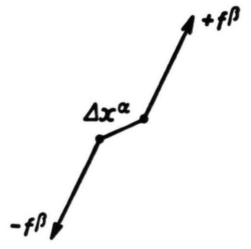

Fig. 1. An elastic dipole.

Now take a step familiar from continuum mechanics: Smooth out these hyperforces into a continuous distribution represented by the "hyperstress" tensor $\Delta^{\alpha\beta\gamma}$. This corresponds, in the above description of a crystal, to a continuous distribution of point defects (compare Kröner's "extra matter" [7, 8]; see also Bilby et al. [9], Anthony [10]). Consider a surface element $\mathrm{d}A_\gamma$. The hyperforce acting through that element is then

$$\mathrm{d}D^{\alpha\beta} = \Delta^{\alpha\beta\gamma}\, \mathrm{d}A_\gamma \,, \quad (2)$$

just as the force acting through $\mathrm{d}A_\gamma$ is given by $\mathrm{d}f^\alpha = \Sigma^{\alpha\gamma}\, \mathrm{d}A_\gamma$, where $\Sigma^{\alpha\gamma}$ is the ordinary (force) stress tensor.

We shall now try to picture the action of hyperstress in 3 dimensions. First, we decompose $\Delta^{\alpha\beta\gamma}$ into its antisymmetric part, its trace, and its symmetric traceless part with respect to its first two indices α and β. (This corresponds to the similar decomposition of the hyperforce $\mathrm{d}D^{\alpha\beta}$.) We obtain ($n$ = number of dimensions, here 3)

$$\Delta^{\alpha\beta\gamma} = \tau^{\alpha\beta\gamma} + (g^{\alpha\beta}/n)\, \Delta^\gamma + \bar{\Delta}^{\alpha\beta\gamma} \,. \quad (3)$$

The antisymmetric part $\tau^{\alpha\beta\gamma} := \Delta^{[\alpha\beta]\gamma}$ could be called rotatory hyperstress, its conventional name is spin moment stress (or torque stress). The symmetric part $\Delta^{(\alpha\beta)\gamma}$ is the proper hyperstress (without moment), it can be resolved into the dilatational hyperstress $\Delta^\gamma := \Delta_\alpha{}^{\cdot\alpha\gamma}$ and the traceless proper hyperstress $\bar{\Delta}^{\alpha\gamma\beta}$ with $\bar{\Delta}_\alpha{}^{\cdot\alpha\gamma} \equiv 0$; $\bar{\Delta}^{[\alpha\beta]\gamma} \equiv 0$. This may be compared with the highly suggestive Fig. 2 in Mindlin [11].

As we might expect, a usual classical point continuum cannot bear hyperstress. Only if we allow for extra degrees of freedom (independent rotations, for example) which characterize so-called polar continua [3–6], can we appreciate the nature of these stresses.

Briefly summarized, a polar continuum is a classical point continuum to each point of which is attached a triad of vectors. Such mathematical continua have been used, for instance, in dislocation theory, in the description of molecular crystals with rotational degrees of freedom and in the description of liquid crystals and anisotropic fluids. In geometry, these structures are principal fibre bundles of the general linear group GL(3,R) or its unimodular subgroup O(3,R) over a base space X_3. (Incidentally, a 4-dimensional continuum of this type with group O(3,1) was used by Einstein and others for the spacetime of his teleparallelism theory.)

Roughly, without going into too much detail, we can say that the ordinary stress tensor $\Sigma^{\alpha\beta}$ deforms the underlying point continuum, and that the hyperstress tensor $\Delta^{\alpha\beta\gamma}$ deforms the triads attached to the points. The action of stress and hyperstress are not unrelated. The generalized conditions of equilibrium (conservation laws), patterned after polar continuum mechanics, are of the form [see, e. g., Mindlin [11], Eq. (4.1)]

$$\nabla_\gamma \Sigma_\alpha{}^{\cdot\gamma} = 0 ; \quad \nabla_\gamma \Delta^{\alpha\beta\gamma} + \Theta^{\alpha\beta} = \Sigma^{\alpha\beta} \,, \quad (4), (5)$$

where $\Theta^{\alpha\beta}$ is a symmetric stress tensor. The antisymmetric part of Eq. (5), $\nabla_\gamma \tau^{\alpha\beta\gamma} = \Sigma^{[\alpha\beta]}$ is the well-known equilibrium condition for moments in continuum mechanics. We can formulate the equilibrium conditions (4, 5) with precision only after we have specified the continuum in detail. We shall do so for a spacetime continuum in Part II. The price of introducing hypermomentum is a more complicated continuum. More intricate statics requires a correspondingly more intricate geometrical description.

3. Momentum and Hypermomentum of Matter in a 4-Dimensional Spacetime Continuum

The concepts of the previous section will now be generalized to spacetime. The analogy between the 3-stress tensor $\Sigma^{\alpha\beta}$ of continuum mechanics and the 4-momentum current (stress-energy or energy-momentum) Σ^{ij} of spacetime has been known since the early days of special relativity. The relativistic generalization of the 3-spin moment stress $\tau^{\alpha\beta\gamma}$ to the 4-dimensional spin angular momentum current τ^{ijk} is also well established [12] and is fundamental to the introduction of spin and torsion into GR (Refs. [13] and [14], and the literature cited therein). The equilibrium conditions of continuum mechanics for forces and moments find their appropriate generalization in the conservation laws of momentum and angular momentum.

We now introduce a working hypothesis: In addition to the 16 components of Σ^{ij}, matter may in general also possess all 64 components of hypermomentum Δ^{ijk}. These quantities satisfy conservation of the type

$$\nabla_k \Sigma_i^{\cdot k} = 0; \quad \nabla_k \Delta^{ijk} + \Theta^{ij} = \Sigma^{ij}, \qquad (6), (7)$$

where Θ^{ij} is a symmetric energy-momentum tensor. The antisymmetric part of (7) is the usual angular momentum conservation law, provided that we identify the spin (24 independent components) by $\tau^{ijk} := \Delta^{[ij]k}$ in analogy to Equation (3). We can also decompose in similar fashion the proper hypermomentum. The dilatation hypermomentum $\Delta^k := \Delta_l^{\cdot lk}$ and the traceless proper hypermomentum $\bar{\Delta}^{ijk}$ arise as $4+36$ new currents. We express this symbolically:

$$\begin{aligned} \text{hypermomentum} \sim\ & \text{spin angular momentum} \\ \oplus\ & \text{dilatation hypermomentum} \\ \oplus\ & \text{traceless proper hypermomentum.} \end{aligned} \qquad (8)$$

The trace Δ^k of the proper hypermomentum has already been encountered in physics! According to Eq. (7), the corresponding conservation law is $\nabla_k \Delta^k + \Theta_k^{\cdot k} = \Sigma_k^{\cdot k}$. We recognize here the divergence relation for the intrinsic part of the so-called dilatation current, provided that Σ^{ij} is identified with the canonical energy-momentum tensor and $\Theta_k^{\cdot k}$ with the divergence of the total dilatation current (compare Jackiw [15] and Ferrara, Gatto, and Grillo [16]). That this identification is consistent will be shown in Part III. Furthermore, we can see from the divergence relation that Δ^k is conserved only if the trace of $(\Sigma - \Theta)_{ij}$ vanishes, that is in the "scaling limit" in elementary particle physics.

The conservation laws for the dilatation current and for the angular momentum current confirm part of our working hypothesis. What remains is to demonstrate the existence of the remaining invariant part of hypermomentum, namely the traceless proper hypermomentum

$$\bar{\Delta}^{ijk} := \Delta^{(ij)k} - \tfrac{1}{4}\, g^{ij} \Delta^k . \qquad (9)$$

This should be possible all the more since the dilatation hypermomentum, $\Delta^k = g_{ij} \Delta^{(ij)k}$, has already been constructed from the symmetric part of hypermomentum. At worst, $\bar{\Delta}^{ijk}$ could vanish. But even then, we would gain an understanding of the reasons for its vanishing. We shall inquire into these questions and seek the new currents $\bar{\Delta}^{ijk}$ in Part III of these notes.

In Sect. 2 we pointed out that we consider hyperstresses (and thus hypermomentum) to be intrinsic quantities which cannot be reduced to anything more fundamental. This is a reasonable assumption in classical field theory. One should keep in mind that, according to the representation theory of the Poincaré group, the antisymmetric part of hypermomentum, spin, is indeed intrinsic. The total angular momentum is the sum of orbital and intrinsic (spin) parts. Similarly, the dilatation current has both an orbital and an intrinsic part [see Ref. [16], Eq. (2.38)]. This can also be seen from considering the conformal group. Therefore we do not doubt the intrinsic nature of hypermomentum as well. The existence of the Belinfante-Rosenfeld [17, 18] and Callan-Coleman-Jackiw [19] procedures for transforming the intrinsic spin and the intrinsic dilatation current into orbital quantities, equivalent upon integration to the original quantities, does not contradict this assertion.

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